

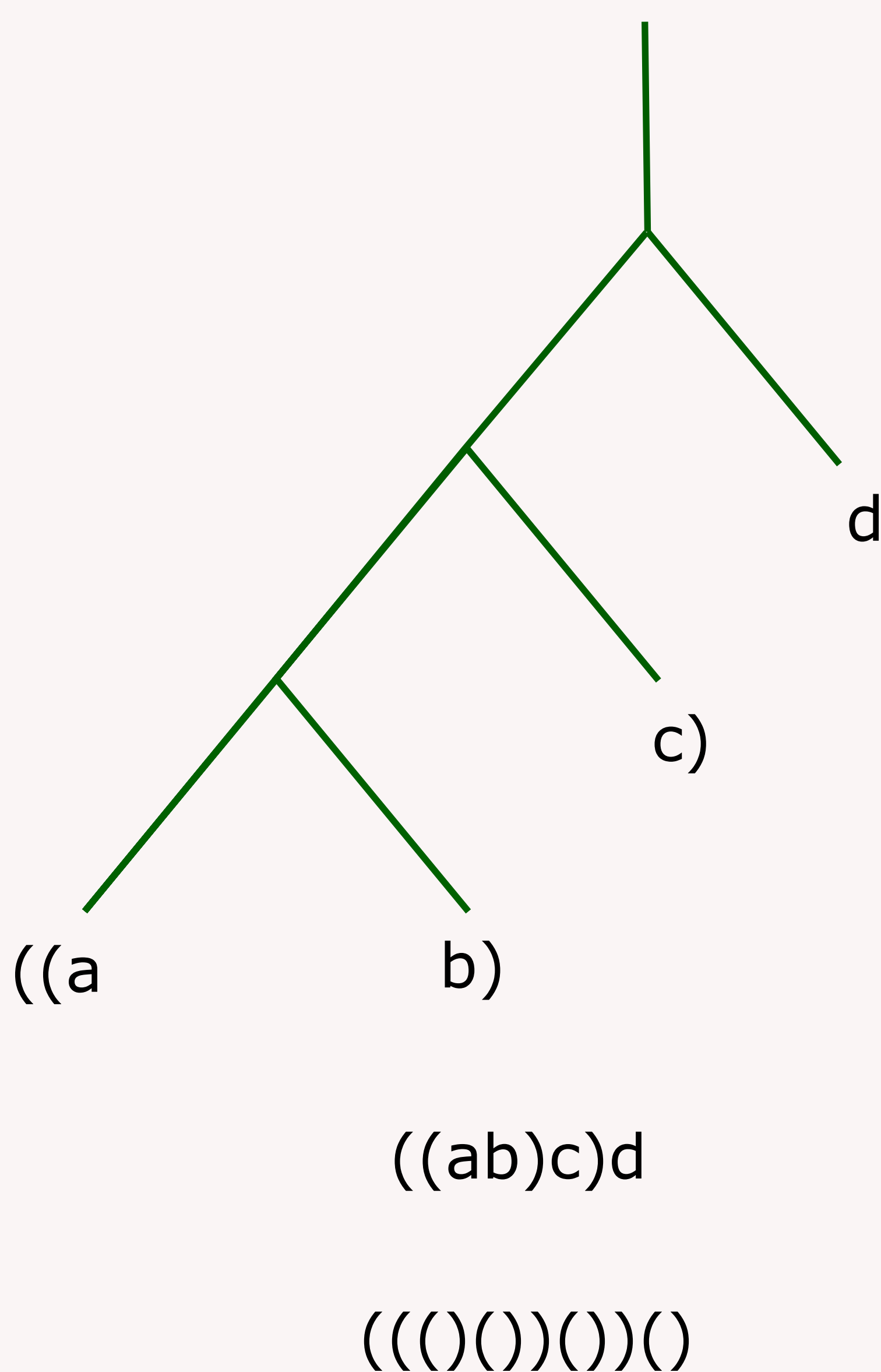
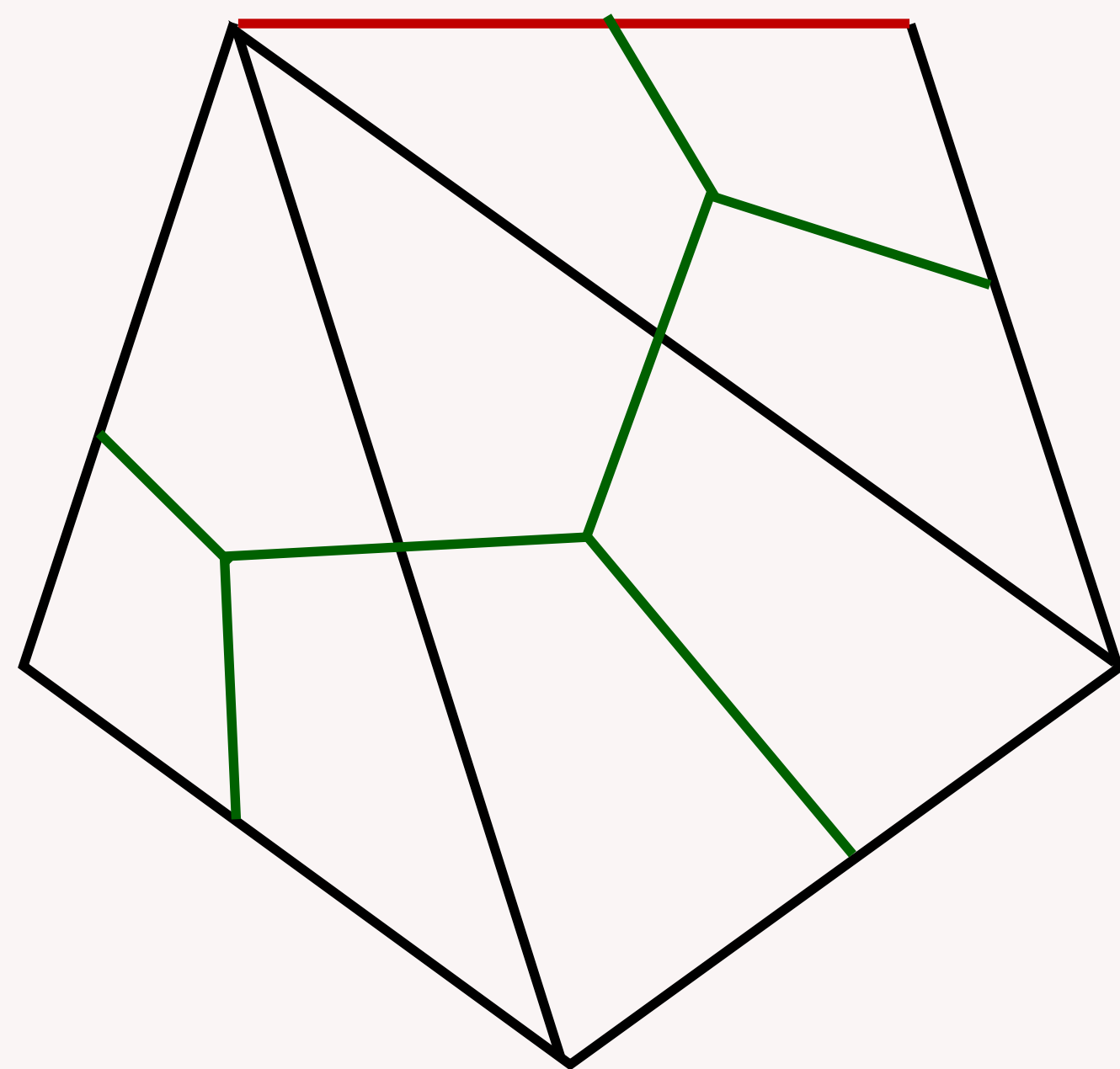
Combinatorics and the Associahedron

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Introduction

The associahedra are a family of polytopes that seem to appear randomly in many areas of mathematics. In this presentation, we will introduce these objects and glimpse how their structure is tied to algebraic and combinatorial problems in a way that makes associahedra a universal object in mathematics.

Equivalent Models



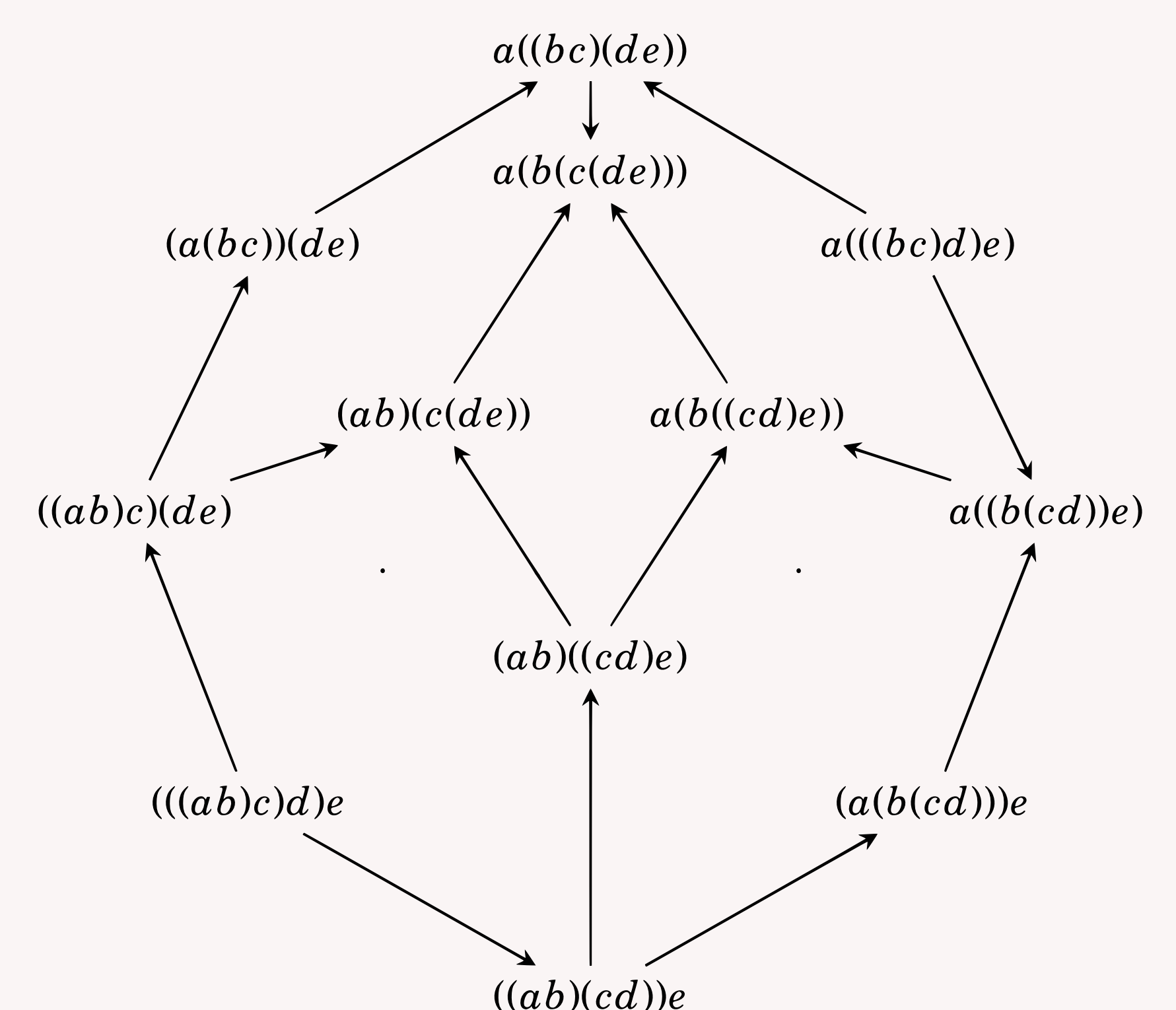
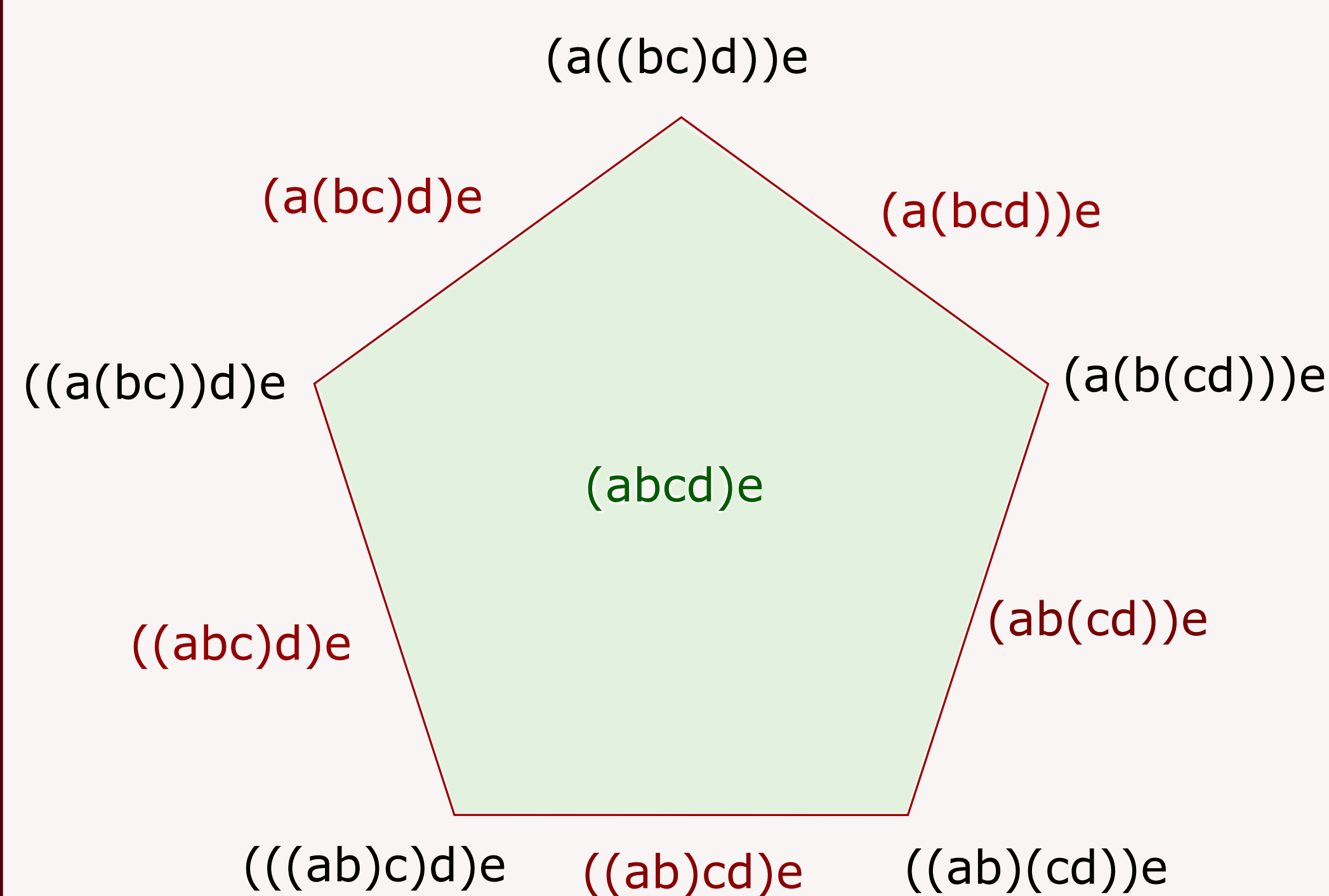
The Associahedron

The associahedron K_n is a convex polytope whose **vertices** represent all binary bracketings of a word of length n , reflecting the different ways of applying the associativity property $(ab)c = a(bc)$. These bracketings are in bijective correspondence with triangulations of an $(n+1)$ -gon, rooted binary trees with n leaves, and stack orderable permutations as defined in [3].

We connect two vertices by an **edge** whenever a single use of the associative law relates two binary bracketings. Thinking of vertices as triangulations, one connects two different triangulations if they are related by a diagonal flip. One can similarly reinterpret these conditions in other models for the edges.

The **higher-dimensional faces** of K_n correspond to partial bracketings, which are attached to the lower-dimensional faces whose bracketings are a refinement of our partial bracketing. Particularly, notice that the **dimension** of K_n is $n - 2$.

The following is an illustration from a face of K_5 (left), and the polyhedron (right). Some of the faces are hidden in the back of the picture, and there is a total of nine faces.



J.-L. Loday proved that the combinatorial polytope described here can be realized as a convex polytope in \mathbb{R}^{n-2} (see [1] for a proof).

Catalan and Schröder Numbers

The number of vertices of K_n is precisely the **Catalan number** C_{n-2} . Using any of the models for the vertices described above, one can find the following recurrence relation:

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

We can find a closed expression for C_n by looking at its **generating function** $C(t) = \sum_{n \geq 0} C_n t^n$. The recurrence relation implies that $C(t)$ verifies the quadratic equation:

$$C(t) = 1 + tC(t)^2 \Rightarrow C(t) = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

A bit of algebra shows that $C_n = \frac{1}{n+1} \binom{2n}{n}$. The first values of Catalan Numbers are 1, 1, 2, 5, 14, 42, 132... For more information on these numbers, see [2].

If one counts the number of facets of all dimensions, one gets to the notion of the Schröder-Hipparchus numbers S_n . Geometrically, these numbers count the number of ways of subdividing an $(n+1)$ -sided polygon by using diagonals (however many). The generating function $S(t) = \sum_{n \geq 0} S_n t^n$ verifies that:

$$S(t) = 1 + tS(t) + tS(t)^2 \Rightarrow S(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}.$$

A more complicated computation leads to:

$$S_n = \sum_{i=1}^n \frac{1}{n} \binom{n}{i} \binom{n}{i-1} 2^{i-1}.$$

References

- [1] E. Ghys, *A Singular Mathematical Promenade*.
- [2] R. Stanley, *Catalan Numbers*
- [3] D. Knuth, *The art of computer programming*